

# Gravothermal Catastrophe and Tsallis' Generalized Entropy of Self-Gravitating Systems III. quasi-equilibrium structure using normalized $q$ -values

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## Abstract

We revisit the issues on the thermodynamic property of stellar self-gravitating system arising from Tsallis' non-extensive entropy. Previous papers (Physica A 307 (2002) 185; *ibid.* (2003) in press (cond-mat/0204315)) have revealed that the extremum-state of Tsallis entropy, so-called *stellar polytrope*, has consistent thermodynamic structure, which predicts the thermodynamic instability due to the negative specific heat. However, their analyses heavily relies on the old Tsallis formalism using standard linear mean values. In this paper, extending our previous study, we focus on the quasi-equilibrium structure based on the standard framework by means of the normalized  $q$ -expectation values. It then turns out that the new extremum-state of Tsallis entropy essentially remains unchanged from the previous result, i.e., the stellar quasi-equilibrium distribution can be described by the stellar polytrope. While the thermodynamic stability for a system confined in an adiabatic wall completely agrees with the previous study and thereby the stability/instability criterion remains unchanged, the stability analysis reveals a new equilibrium property for the system surrounded by a thermal bath. In any case, the stability/instability criteria are consistently explained from the presence of negative specific heat and within the formalism, the stellar polytrope is characterized as a plausible non-extensive meta-equilibrium state.

*Key words:* non-extensive entropy, self-gravitating system, gravothermal instability, negative specific heat, stellar polytrope

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## 1 Introduction

In many astrophysical problems involving self-gravitating many-body system, thermodynamic or statistical mechanical treatment usually loses its power due to the peculiar feature of long-range attractive force. In some restricted cases with the long-term evolution of stellar self-gravitating systems, however, thermodynamic stability criterion or statistical mechanical analysis recovers the physical relevance and plays an essential role in predicting the fate of such system. In fact, it is well-known that late-time stellar dynamical evolution of the globular clusters as a real astrophysical system is driven by so-called *gravothermal catastrophe*, i.e., thermodynamic instability arising from the negative specific heat, which is widely accepted as a fundamental astrophysical process [1,2,3]. Historically, the gravothermal catastrophe has been investigated in detail considering the very idealized situation, i.e., stellar self-gravitating system consisting of many particles confined within a cavity of hard sphere [4,5,6,7,8]. In particular, special attention to the statistical mechanical approach using the Boltzmann-Gibbs entropy has been paid.

Recently, this classic issue was re-analyzed by present authors on the basis of the non-extensive thermostatistics with Tsallis' generalized entropy [9] (for comprehensive review of Tsallis' non-extensive formalism and its application to the other subject of physics, see Refs.[10,11]). In contrast to the Boltzmann-Gibbs entropy, the quasi-equilibrium distribution<sup>1</sup>

characterized by the Tsallis entropy can be reduced to the stellar polytropic system[12,13]. Evaluating the second variation of entropy around the quasi-equilibrium state, we have developed the stability analysis and discussed the criterion for onset of gravothermal catastrophe (Ref.[14], hereafter paper I). Further, to clarify the origin of this instability, thermodynamic properties has been investigated in detail calculating the specific heat of the stellar polytrope (Ref.[15], hereafter paper II). The most noticeable thing in their papers is that the existence of thermodynamic instability indicated from the second variation of entropy or free-energy is completely explained from the presence of negative specific heat. As a consequence, the gravothermal instability appears at the polytrope indices  $n > 5$  for a system confined in an adiabatic wall and at  $n > 3$  for a system surrounded by a thermal bath.

While the results in papers I and II are indeed satisfactory and the physical interpretation of the instability is fully consistent with previous early

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<sup>1</sup> Here and in what follows, we use the term *quasi-equilibrium* rather than the (thermal) equilibrium, since there is no strict thermal equilibrium in a self-gravitating system. The quasi-equilibrium state means that the system is, at least, stable in a dynamical equilibrium state.

works, from a standard viewpoint of the Tsallis non-extensive formalism, several important issues still remain unresolved. Among these, the most crucial problem is the choice of the statistical average in non-extensive thermostatics. The previous analyses have been all investigated utilizing the *old* Tsallis formalism with the standard linear mean values, however, a more sophisticated framework by means of the normalized  $q$ -values has been recently presented [16,17]. As several authors advocated, the analysis using normalized  $q$ -values is thought to be essential, since the undesirable divergences in some physical systems can be eliminated safely when introducing the normalized  $q$ -values. Further, non-uniqueness of the Boltzmann-Gibbs theory has been shown using the normalized  $q$ -values [18]. Of course, this does not imply that all the analyses with standard linear means or un-normalized  $q$ -values lose the physical significance<sup>2</sup>, however, in order to pursue the physical reality of the non-extensive thermodynamics, thermodynamic structure of the stellar self-gravitating system still needs to be investigated and the relation between the old and the new formalism must be clarified.

This paper especially focuses on this matter starting from the construction of the extremum-entropy state by means of the normalized  $q$ -values. Within a mean-field treatment, we investigate the thermodynamic property of the quasi-equilibrium distribution surrounded by an adiabatic and a thermally conducting wall. In particular, we discuss the existence or the absence of thermodynamic instability evaluating the specific heat of the quasi-equilibrium systems. Further, in order to check the consistency of the thermodynamic structure, the stability/instability criteria obtained from the thermodynamic property are re-analyzed from the second variation of entropy and free-energy. We found that the extremum state of the Tsallis entropy essentially remains unchanged and is described by the stellar polytrope. While the stability of the system in an adiabatic wall exactly coincides with the results in paper I, the new formalism using the normalized  $q$ -values reveals a new thermodynamic structure for a stellar system surrounded by a thermal bath. This point will be discussed in detail comparing it with the previous results.

The paper is organized as follows. In Section 2, employing the standard Tsallis formalism, we revisit the issue on the most probable state of stellar quasi-equilibrium distribution on the basis of maximum entropy principle. While the resultant extremum state of Tsallis entropy reduces to the same stellar polytropic distribution as previously found, the new equilibrium distribution has several distinct structures. Taking fully account of this fact, in Section 3, the thermodynamic properties of stellar polytropic system are investigated in detail. Thermodynamic temperature in stellar system is identified through the *modified* Clausius relation, which is indeed consistent with the recent claim

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<sup>2</sup> Indeed, even the old formalism consistently recovers the standard Legendre transform structure leading to the usual thermodynamic relations [19,20].

based on the thermodynamic zero-th law. Then, we evaluate the specific heat and discuss the existence or absence of thermodynamic instability. In Section 4, the stability/instability criteria are re-considered by means of the second variation of entropy and free-energy. In contrast to the previous analysis using the old Tsallis formalism, the second variation of free-energy shows a distinct thermodynamic structure. Nevertheless, the zero-eigenvalue problem in each case exactly recovers the marginal stability condition inferred from the specific heat. Thus, within the new formalism using normalized  $q$ -values, all the analyses are consistent and the stellar polytrope can be regarded as a plausible non-extensive thermal state. Finally, Section 5 is devoted to the conclusion and the discussion.

## 2 Maximum entropy principle revisited

Throughout the paper, we pursue to investigate the equilibrium property of the stellar self-gravitating system consisting of  $N$  particles confined in a spherical cavity of radius  $r_e$ . For simplicity, each particle has the same mass  $m_0$  and interacts via Newton gravity only. Then, the total mass becomes  $M = Nm_0$ . In this situation, owing to the maximum entropy principle, we seek the most probable quasi-equilibrium distribution in an adiabatic treatment. That is, we consider the quasi-equilibrium structure as the extremum-entropy state in which the particles elastically bounce from the wall, keeping the mass  $M$  and energy  $E$  constant.

Following the papers I and II, we treat this issue employing the mean-field approach that the correlation between particles is smeared out and the system can be simply described by the one-particle distribution function  $f(\mathbf{x}, \mathbf{v})$ , defined in six-dimensional phase-space  $(\mathbf{x}, \mathbf{v})$ . In this treatment, the one-particle distribution is regarded as a fundamental quantity characterizing the stellar system. Let us denote the phase-space element as  $h^3 = (l_0 v_0)^3$  with unit length  $l_0$  and unit velocity  $v_0$  and define the integral measure  $d^6\boldsymbol{\tau} \equiv d^3\mathbf{x}d^3\mathbf{v}/h^3$ . Regarding the function  $f(\mathbf{x}, \mathbf{v})$  as a fundamental statistical quantity, the energy and the mass are respectively expressed as follows:

$$E = K + U = \int \left\{ \frac{1}{2} v^2 + \frac{1}{2} \Phi(\mathbf{x}) \right\} f(\mathbf{x}, \mathbf{v}) d^6\boldsymbol{\tau}, \quad (1)$$

$$M = \int f(\mathbf{x}, \mathbf{v}) d^6\boldsymbol{\tau}, \quad (2)$$

where the quantity  $\Phi(\mathbf{x})$  is the gravitational potential given by

$$\Phi(\mathbf{x}) = -G \int \frac{f(\mathbf{x}', \mathbf{v}')}{|\mathbf{x} - \mathbf{x}'|} d^6 \boldsymbol{\tau}'. \quad (3)$$

On the other hand, in the new framework of Tsallis' non-extensive thermostatics, all the macroscopic observables of the quasi-equilibrium system can be characterized by the escort distribution, but the escort distribution itself is not thought to be fundamental. Rather, there exists a more fundamental probability function  $p(\mathbf{x}, \mathbf{v})$  that quantifies the phase-space structure. With a help of this function, the escort distribution is defined and the macroscopic observables are expressed as the normalized  $q$ -expectation value as follows (e.g., Refs.[16,17]):

$$\text{escort distribution} : P_q(\mathbf{x}, \mathbf{v}) = \frac{\{p(\mathbf{x}, \mathbf{v})\}^q}{\int d^6 \boldsymbol{\tau} \{p(\mathbf{x}, \mathbf{v})\}^q}, \quad (4)$$

$$\text{normalized } q\text{-value} : \langle O_i \rangle_q = \int d^6 \boldsymbol{\tau} O_i P_q(\mathbf{x}, \mathbf{v}). \quad (5)$$

And based on the fundamental probability  $p(\mathbf{x}, \mathbf{v})$ , the Tsallis entropy is given by

$$S_q = -\frac{1}{q-1} \int d^6 \boldsymbol{\tau} [\{p(\mathbf{x}, \mathbf{v})\}^q - p(\mathbf{x}, \mathbf{v})]. \quad (6)$$

Note that the probability  $p(\mathbf{x}, \mathbf{v})$  satisfies the normalization condition:

$$\int d^6 \boldsymbol{\tau} p(\mathbf{x}, \mathbf{v}) = 1. \quad (7)$$

To apply the above Tsallis formalism to the present problem without changing the definition of energy and mass (1) and (2), we identify the one-particle distribution with the escort distribution  $P_q$ , not the probability function  $p(\mathbf{x}, \mathbf{v})$ :

$$f(\mathbf{x}, \mathbf{v}) = M \frac{\{p(\mathbf{x}, \mathbf{v})\}^q}{N_q}; \quad N_q = \int d^6 \boldsymbol{\tau} \{p(\mathbf{x}, \mathbf{v})\}^q \quad (8)$$

so as to satisfy the mass conservation (2). With this identification (8), the normalized  $q$ -values are naturally incorporated into our mean-field treatment. As a consequence, later analyses become almost parallel to the previous study, in a form-invariant manner.

Now, adopting the relation (8), let us seek the extremum-entropy state under the constraints (1) and (7). As has been discussed recently, there are two approaches that extremize the entropy under certain constraints, i.e., the method

developed by Tsallis, Mendes & Plastino (TMP)[16] and the optimal Lagrange multiplier(OLM) method by Martínez et al. [17]. Here, we specifically apply the TMP procedure and find the extremum entropy state. The alternative derivation using OLM method is presented in Appendix A. The only differences in the final expressions between the TMP and the OLM method are the dependence of the quantity  $N_q$ , which can be summarized in a unified form (see Eq.(15) with (16)).

The variational problem in the TMP method is given by the following equation:

$$\delta \left[ S_q - \alpha \left\{ \int d^6 \boldsymbol{\tau} p - 1 \right\} - \beta \left\{ \int d^6 \boldsymbol{\tau} \left( \frac{1}{2} v^2 + \frac{1}{2} \Phi \right) f - E \right\} \right] = 0, \quad (9)$$

where the variables  $\alpha$  and  $\beta$  denote the Lagrange multipliers. The variation with respect to the probability  $p(\mathbf{x}, \mathbf{v})$  leads to

$$\int d^6 \boldsymbol{\tau} \left[ -\frac{1}{q-1} (q p^{q-1} - 1) \delta p - \alpha \delta p - \beta \left( \frac{1}{2} v^2 + \Phi \right) \delta f \right] = 0. \quad (10)$$

Here, we used the fact that  $\int d^6 \boldsymbol{\tau} \delta \Phi f = \int d^6 \boldsymbol{\tau} \Phi \delta f$ . The above equation includes the variation  $\delta f$ , which can be expressed with a help of the relation (8):

$$\delta f(\mathbf{x}, \mathbf{v}) = q f(\mathbf{x}, \mathbf{v}) \left\{ \frac{\delta p}{p(\mathbf{x}, \mathbf{v})} - \frac{1}{M} \int d^6 \boldsymbol{\tau}' f(\mathbf{x}', \mathbf{v}') \frac{\delta p'(\mathbf{x}', \mathbf{v}')}{p(\mathbf{x}', \mathbf{v}')} \right\}. \quad (11)$$

Then, substituting the above expression into (10) becomes

$$\int d^6 \boldsymbol{\tau} \left[ -\frac{1}{q-1} (q p^{q-1} - 1) - \alpha - \beta M q \frac{p^{q-1}}{N_q} \left( \frac{1}{2} v^2 + \Phi - \varepsilon \right) \right] \delta p = 0. \quad (12)$$

with the quantity  $\varepsilon$  being

$$\varepsilon = \frac{1}{M} \int d^6 \boldsymbol{\tau} \left( \frac{1}{2} v^2 + \Phi \right) f. \quad (13)$$

In arriving at the equation (12), we have exchanged the role of the variables  $(\mathbf{x}, \mathbf{v}) \leftrightarrow (\mathbf{x}', \mathbf{v}')$ . Since the above equation must be satisfied independently of

the choice of the variation  $\delta p$ , we obtain

$$-\frac{1}{q-1} \left( q p^{q-1} - 1 \right) - \alpha - \beta M q \frac{p^{q-1}}{N_q} \left( \frac{1}{2} v^2 + \Phi - \varepsilon \right) = 0, \quad (14)$$

which reduces to the same power-law distribution of  $\varepsilon$  as has been derived in papers I and II. That is, even in the new formalism, the extremum-entropy state remains unchanged and can be described by the so-called *stellar polytrope*. Together with the result by OLM method (Appendix A), it can be summarized as follows:

$$f(x, v) = M \frac{\{p(x, v)\}^q}{N_q} = A \left[ \Phi_0 - \frac{1}{2} v^2 - \Phi(x) \right]^{q/(1-q)}, \quad (15)$$

where we define the constants  $A$  and  $\Phi_0$ :

$$A = \frac{M}{N_q} \left\{ \frac{q(1-q)}{\alpha(1-q)+1} \frac{\beta M}{N'_q} \right\}^{q/(1-q)}, \quad \Phi_0 = \frac{N'_q}{\beta M(1-q)} + \varepsilon. \quad (16)$$

Note that the quantity  $N'_q$  means  $N'_q = N_q$  for the TMP method and  $N'_q = 1$  for the result using OLM procedure.

For later analysis, we define the density  $\rho(r)$  and the isotropic pressure  $P(r)$  at the radius  $r = |\mathbf{x}|$  as

$$\begin{aligned} \rho(r) &\equiv \int \frac{d^3 \mathbf{v}}{h^3} f(\mathbf{x}, \mathbf{v}), \\ &= 4\sqrt{2}\pi B \left( \frac{3}{2}, \frac{1}{1-q} \right) \frac{A}{h^3} [\Phi_0 - \Phi(r)]^{1/(1-q)+1/2} \end{aligned} \quad (17)$$

and

$$\begin{aligned} P(r) &\equiv \int \frac{d^3 \mathbf{v}}{h^3} \frac{1}{3} v^2 f(\mathbf{x}, \mathbf{v}), \\ &= \frac{8\sqrt{2}\pi}{3} B \left( \frac{5}{2}, \frac{1}{1-q} \right) \frac{A}{h^3} [\Phi_0 - \Phi(r)]^{1/(1-q)+3/2}. \end{aligned} \quad (18)$$

Here, the function  $B(a, b)$  denotes the beta function. These two equations lead to the following polytropic relation:

$$P(r) = K_n \rho^{1+1/n}(r), \quad (19)$$

with the polytrope index  $n$  given by

$$n = \frac{1}{1-q} + \frac{1}{2}, \quad (20)$$

and with the dimensional constant  $K_n$ :

$$K_n = \frac{1}{n+1} \left[ \frac{4\sqrt{2}\pi}{h^3} B\left(\frac{3}{2}, n - \frac{1}{2}\right) \left\{ \frac{q(1-q)}{\alpha(1-q)+1} \frac{\beta M}{N'_q} \right\}^{n-3/2} \frac{M}{N_q} \right]^{-1/n}. \quad (21)$$

Using these quantities, the one-particle distribution can be rewritten as follows:

$$f(x, v) = \frac{1}{4\sqrt{2}\pi B(3/2, n - 1/2)} \frac{\rho h^3}{\{(n+1) K_n \rho^{1/n}\}^{3/2}} \times \left\{ 1 - \frac{v^2/2}{(n+1) K_n \rho^{1/n}} \right\}^{n-3/2}, \quad (22)$$

which agrees with the previous result (see Eq.(16) with the identification  $(n+1)K_n = (n-3/2)T$  in paper I).

While the resultant form of the quasi-equilibrium distribution (22) turns out to be invariant irrespective of the choice of the statistical averages, we should be aware of the two important differences between the old and the new Tsallis formalism, which we shall describe below.

First, the relation between the polytrope index  $n$  and Tsallis parameter  $q$  in the polytropic relation (19) differs from the one obtained previously, but is related to it through the *duality transformation*,  $q \leftrightarrow 1/q$  (see Eq.(14) in paper I or Eq.(12) in paper II). This property has been first addressed in Ref.[16] in more general context, together with the changes in Lagrangian multiplier  $\beta$ . The duality relation implies that all of the thermodynamic properties in the new formalism can also be translated into those obtained in the old formalism. As shown in section 3 and 4, this is indeed true in the system confined in an adiabatic wall(micro-canonical ensemble case), however, the duality of thermodynamic structure cannot hold in the system surrounded by a thermal bath(canonical ensemble case).

Second, notice that the quasi-equilibrium distribution (15) with (16) contains the new quantities  $N_q$  and  $\varepsilon$ , which implicitly depend on the distribution function itself. In marked contrast to the result in old Tsallis formalism, this fact gives rise to the non-trivial thermodynamic relations as follows. Using the definitions of density and pressure (17) and (18), the quantity  $\varepsilon$  becomes



$$\begin{aligned}
\varepsilon &= \frac{1}{M} \left\{ \frac{3}{2} \int d^3x P(x) + \int d^3x \rho(x) \Phi(x) \right\} \\
&= \frac{1}{M} \left\{ \frac{3}{2} \int d^3x P(x) - \int d^3x \rho(x) [\Phi_0 - \Phi(x)] \right\} + \Phi_0.
\end{aligned}$$

Further using the relation  $\Phi_0 - \Phi(x) = (n+1)(P/\rho)$  from (17) and (18) and substituting the equation (16) into the above expression, the variable  $\varepsilon$  is cancelled and the equation reduces to

$$\frac{N'_q}{\beta} = \int d^3x P(x). \quad (23)$$

As for the dependence of  $N_q$ , the normalization condition (7) implies that

$$N_q = \left[ \int d^6\boldsymbol{\tau} \{ f(\boldsymbol{x}, \boldsymbol{v}) \}^{1/q} \right]^{-q}.$$

Substituting the distribution function (22) into the above equation and integrating over the velocity space, after some manipulation, one obtains

$$N_q = c_n K_n^{(3/2)/(n-1/2)} \left\{ \int d^3x \rho^{1+1/n}(x) \right\}^{-(n-3/2)/(n-1/2)} \quad (24)$$

with the constant  $c_n$  given by

$$c_n = \frac{4\sqrt{2}\pi B(3/2, n-1/2)}{\left\{ 4\sqrt{2}\pi B(3/2, n+1/2) \right\}^{(n-3/2)/(n-1/2)}} \left\{ \frac{(n+1)^{3/2}}{h^3} \right\}^{1/(n-1/2)}. \quad (25)$$

Equations (23) and (24) play a crucial role in determining the thermodynamic temperature of stellar polytrope in section 3, as well as the onset of gravothermal instability in section 4. In particular, the equation (23) yields the *radius-mass-temperature relation* characterizing the quasi-equilibrium structure of the system confined in a thermal bath.

Keeping the above remarks in mind, hereafter, we will specifically focus on the spherically symmetric case with the polytrope index  $n > 3/2$  ( $q > 0$ ), in which the quasi-equilibrium distribution is at least dynamically stable (see Chap.5 of Ref.[1]). In this case, despite the above detailed differences, the stellar quasi-equilibrium distribution can be characterized by the so-called *Emden solutions* (e.g., [21,22]) and all the physical quantities are expressed in terms of the homology invariant variables  $(u, v)$ , which are subsequently used in later analysis. In Appendix B, together with some useful integral formulae, we summarize the relation between the Emden solution and the stellar polytropic distribution.

### 3 Thermodynamic properties of stellar polytrope

Having established the quasi-equilibrium distribution, we now investigate the thermodynamic properties of stellar polytropic system. To do this, we adopt the same procedure as in paper II. That is, we examine the Clausius relation under the quasi-static variation for the new equilibrium system in section 3.1. In contrast to the previous analysis, the thermodynamic temperature can be identified with a help of (23), consistently with the recent claim based on the thermodynamic zero-th law. Using this temperature, in section 3.2, presence or absence of thermodynamic instability is discussed evaluating the specific heat of stellar system.

#### 3.1 On the definition of thermodynamic temperature

As usual, the thermodynamic instability in stellar system is intimately related to the presence of negative specific heat [5,23]. The evaluation of specific heat is thus necessary for clarifying the thermodynamic property. In this regard, the identification of temperature in stellar system is the most essential task.

In the new framework of Tsallis' non-extensive thermostatistics, the physically plausible thermodynamic temperature,  $T_{\text{phys}}$ , can be defined from the zero-th law of thermodynamics [24,25,26]. Then, the thermodynamic temperature in non-extensive system differs from the usual one, i.e., the inverse of the Lagrange multiplier,  $\beta$ . Depending on the methods extremizing the entropy, one has

$$T_{\text{phys}} = \begin{cases} \beta^{-1} & ; \quad (\text{OLM method}), \\ [1 + (1 - q) S_q] \beta^{-1} & ; \quad (\text{TMP method}). \end{cases} \quad (26)$$

Translating the above result into our notation immediately yields that the physical temperature in stellar polytropic system is given by  $T_{\text{phys}} = N'_q/\beta$ . However, the relation (26) should be carefully applied to the present case, since the verification of thermodynamic zero-th law is very difficult in stellar equilibrium system with long-range interaction. Furthermore, even using the new formalism, the energy  $E$  still keeps non-extensive due to the self-referential form of the potential energy (see Eqs.(1)(3)). In order to validate the use of the definition (26), we therefore adopt a rather simple procedure as examined in paper II. That is, we consider the relation between heat transfer and entropy change in the quasi-static treatment under keeping the total mass constant. According to the definition (26), the Clausius relation is appropriately

modified and is expressed as follows [24,25,26]:

$$dS_q = \frac{1}{T_{\text{phys}}} \{1 + (1 - q) S_q\} d'Q. \quad (27)$$

This, in turn, implies that the modified Clausius relation can be used as a consistency check of the physical temperature (26) in stellar system.

Let us first write down the entropy of extremum state:

$$S_q = \left(n - \frac{1}{2}\right) (N_q - 1). \quad (28)$$

From equation (24) with the polytropic relation (19), the quantity  $N_q$  in equation (28) is rewritten as

$$\begin{aligned} N_q &= c_n K_n^{n/(n-1/2)} \left\{ \int d^3x P(x) \right\}^{-(n-3/2)(n-1/2)} \\ &= c_n K_n^{n/(n-1/2)} \left\{ \frac{1}{n-5} \frac{GM^2}{r_e} \left( \frac{n+1}{v_e} - 2 \frac{u_e}{v_e} - 1 \right) \right\}^{-(n-3/2)/(n-1/2)} \end{aligned} \quad (29)$$

with a help of the integral formula (88). Here, the variables  $(u_e, v_e)$  denote the homology invariants evaluated at the boundary  $r = r_e$  (see definitions (84)(85)). Note also the fact that the constant  $K_n$  is expressed in terms of  $(u_e, v_e)$ -variables:

$$K_n = \frac{P_e}{\rho_e^{1+1/n}} = \frac{GM}{r_e} \frac{1}{v_e} \left( \frac{4\pi r_e^3}{M} \frac{1}{u_e} \right)^{1/n}. \quad (30)$$

Using these expressions, the variation of the quantities  $N_q$  and  $K_n$  respectively becomes

$$\begin{aligned} \left(n - \frac{1}{2}\right) \frac{dN_q}{N_q} &= n \frac{dK_n}{K_n} + \left(n - \frac{3}{2}\right) \frac{dr_e}{r_e} \\ &\quad - \frac{n-3/2}{n-5} \frac{GM^2\beta}{r_e N_q'} \left\{ -\frac{n+1}{v_e} \frac{dv_e}{v_e} - 2 \left( \frac{du_e}{u_e} - \frac{dv_e}{v_e} \right) \right\}, \end{aligned} \quad (31)$$

and

$$\frac{dK_n}{K_n} = -\frac{n-3}{n} \frac{dr_e}{r_e} - \frac{dv_e}{v_e} - \frac{1}{n} \frac{du_e}{u_e}. \quad (32)$$

In the last line of equation (31), we have used the relation (23). Collecting these results, the entropy change  $dS_q$  can be expressed as the variations of both the homology invariants  $(u_e, v_e)$  and the wall radius  $r_e$  as follows:

$$dS_q = \left(n - \frac{1}{2}\right) dN_q = N_q \left[ \frac{3}{2} \frac{dr_e}{r_e} - \frac{du_e}{u_e} - n \frac{dv_e}{v_e} - \frac{n - 3/2}{n - 5} \frac{GM^2\beta}{r_e N'_q} \left\{ -\frac{n+1}{v_e} \frac{dv_e}{v_e} - 2 \left( \frac{du_e}{u_e} - \frac{dv_e}{v_e} \right) \right\} \right], \quad (33)$$

In the above equation, the term  $GM^2\beta/(r_e N'_q)$  can be factorized out using the relation (23) and we have

$$dS_q = N_q \frac{GM^2\beta}{r_e N'_q} \left[ -\frac{3}{2} \frac{1}{n-5} \left( 2 \frac{u_e}{v_e} - \frac{n+1}{v_e} + 1 \right) \frac{dr_e}{r_e} + \frac{1}{n-5} \frac{1}{2v_e} \left\{ 4 \left( n - \frac{1}{2} \right) u_e + 2v_e - 2(n+1) \right\} \frac{du_e}{u_e} + \{6u_e + 2nv_e - 3(n+1)\} \frac{dv_e}{v_e} \right]. \quad (34)$$

Further notice the fact that the variation of homology invariants is expressed as the variation of dimensionless quantity  $d\xi_e$  (see Eq.(86)). The entropy change is finally reduced to the following expression:

$$dS_q = N_q \frac{GM^2\beta}{r_e N'_q} \left[ -\frac{3}{2} \frac{1}{n-5} \left( 2 \frac{u_e}{v_e} - \frac{n+1}{v_e} + 1 \right) \frac{dr_e}{r_e} - \frac{n-2}{n-5} \frac{1}{2v_e} \times \left\{ 4u_e^2 + 2u_e v_e - \left( 8 + 3 \frac{n+1}{n-2} \right) u_e - \frac{3}{n-2} v_e + 3 \left( \frac{n+1}{n-2} \right) \right\} \frac{d\xi_e}{\xi_e} \right]. \quad (35)$$

The Clausius relation in a quasi-static variation relates the variation of entropy with the heat change. According to the first-law of thermodynamics, the heat change in a quasi-static variation is estimated as follows (see Eq.(26) in paper II):

$$\begin{aligned} d'Q &= dE + P_e dV = d \left( -\lambda \frac{GM^2}{r_e} \right) + 4\pi r_e^2 P_e dr_e \\ &= \frac{GM^2}{r_e} \left\{ \left( \lambda + \frac{u_e}{v_e} \right) \frac{dr_e}{r_e} - \xi_e \frac{d\lambda}{d\xi_e} \frac{d\xi_e}{\xi_e} \right\}, \end{aligned} \quad (36)$$

where the dimensionless quantity  $\lambda$  related to the energy and its derivative  $d\lambda/d\xi_e$  are respectively given by (see Eq.(91)):

$$\lambda \equiv -\frac{r_e E}{GM^2} = -\frac{1}{n-5} \left\{ \frac{3}{2} \left( 1 - \frac{n+1}{v_e} \right) + (n-2) \frac{u_e}{v_e} \right\}, \quad (37)$$

and

$$\begin{aligned} \xi_e \frac{d\lambda}{d\xi_e} &= \frac{n-2}{n-5} \frac{g(u_e, v_e)}{2v_e}; \\ g(u, v) &= 4u^2 + 2uv - \left\{ 8 + 3 \left( \frac{n+1}{n-2} \right) \right\} u - \frac{3}{n-2} v + 3 \left( \frac{n+1}{n-2} \right). \end{aligned} \quad (38)$$

Hence, the comparison between (35) and (36) with (37)(38) leads to the modified Clausius relation corresponding to the equation (27):

$$dS_q = \frac{\beta}{N'_q} N_q (dE + P_e dV) = \frac{\beta}{N'_q} \{1 + (1-q) S_q\} d'Q. \quad (39)$$

Therefore, with this relation, the thermodynamic temperature can be consistently identified along the line of the argument in Ref.[24] and the plausible physical temperature is now

$$T_{\text{phys}} = \frac{N'_q}{\beta}. \quad (40)$$

### 3.2 Thermodynamic instability arising from the negative specific heat

Once adopting the definition (40), we immediately obtain the *radius-mass-temperature relation* in terms of the homology invariants as follows. Equation (23) in section 2 leads to

$$T_{\text{phys}} = \int d^3x P(x) = -\frac{1}{n-5} \frac{GM^2}{r_e} \left( 2 \frac{u_e}{v_e} - \frac{n+1}{v_e} + 1 \right) \quad (41)$$

from (88). Thus, defining the dimensionless quantity  $\eta = GM^2/(r_e T_{\text{phys}})$ , we obtain

$$\eta \equiv \frac{GM^2}{r_e T_{\text{phys}}} = \frac{(n-5) v_e}{n+1-2u_e-v_e}. \quad (42)$$

This is in marked contrast to the result using the standard linear mean (c.f. Eq.(31) of paper II). While the radius-mass-temperature relation in previous paper includes the residual dimensional parameter  $h = (l_0 v_0)^3$ , the expression (42) has no such parameter dependence and is quite similar to the result in

Boltzmann-Gibbs case (e.g., Eq.(29) in Ref.[5] or Eq.(25) in Ref.[27])). The nice form of the radius-mass-temperature relation implies that the specific heat can be determined independently of the residual parameter  $h$ , which is indeed a desirable property for a rigid theoretical prediction without any uncertainty.

By definition, the specific heat at constant volume  $C_v$  is given by

$$C_v \equiv \left( \frac{dE}{dT_{\text{phys}}} \right)_e = \frac{\left( \frac{dE}{d\xi} \right)_e}{\left( \frac{dT_{\text{phys}}}{d\xi} \right)_e}. \quad (43)$$

The numerator and the denominator in the last expression are respectively rewritten as follows:

$$\left( \frac{dE}{d\xi} \right)_e = -\frac{GM^2}{r_e} \frac{d\lambda}{d\xi_e} = -\frac{GM^2}{r_e} \frac{n-2}{n-5} \frac{g(u_e, v_e)}{2u_e \xi_e}, \quad (44)$$

and

$$\left( \frac{dT_{\text{phys}}}{d\xi} \right)_e = \frac{GM^2}{r_e} \frac{d}{d\xi_e} \eta^{-1} = \frac{GM^2}{r_e} \frac{1}{n-5} \frac{k(u_e, v_e)}{u_e \xi_e}. \quad (45)$$

In the above equations, the functions  $g(u_e, v_e)$  is already given by (38) and the function  $k(u_e, v_e)$  is expressed as

$$k(u_e, v_e) = 4u_e^2 + 2u_e v_e - (n+9)u_e - v_e + n + 1, \quad (46)$$

from (86). Thus, the expression of specific heat can be reduced to the following simple form:

$$C_v = -\frac{n-2}{2} \frac{g(u_e, v_e)}{k(u_e, v_e)}. \quad (47)$$

Provided the homology invariants at the boundary, the specific heat in the new formalism is uniquely determined, free from the residual parameter  $h$  (c.f. Eq.(42) in paper II). In this sense, the result (47) can be regarded as a successful outcome of the new Tsallis formalism using the normalized  $q$ -values.

Now, we focus on the thermodynamic instability inferred from the qualitative behavior of specific heat. Recall that the thermodynamic instability appears when the specific heat of the system changes its sign. From (47), we readily

expect the two possibilities. One is the case when the function  $g(u_e, v_e)$  changes its sign. In this case, the condition for marginal stability,  $C_V = 0$ , becomes

$$g(u_e, v_e) = 0. \quad (48)$$

The other cases appear when the sign of the function  $k(u_e, v_e)$  is changed. In this case, the marginal stability leads to the divergent behavior,  $C_V \rightarrow \pm\infty$  and the condition is given by

$$k(u_e, v_e) = 0. \quad (49)$$

To see how and when the instability develops, in Fig. 1, we plot a family of Emden solutions in the  $(\eta, \lambda)$ -plane. Since the dimensionless parameters  $\lambda$  and  $\eta$  are respectively proportional to  $-E$  and  $T_{\text{phys}}^{-1}$ , the signature of the specific heat can be easily deduced from the slope of the curve. Note that each point along the trajectory represents an Emden solution evaluated at the different value of the radius  $r_e$ . From the boundary condition (83), all the trajectories start from  $(\eta, \lambda) = (0, -\infty)$ , corresponding to the limit  $r_e \rightarrow 0$ . As increasing the radius, the trajectories first move to the upper-right direction as marked by the arrow, and they suddenly change their direction to upper-left. This means that the divergent behavior of specific heat eventually appears and beyond that point, the signature of specific heat changes from positive to negative. That is, the potential energy dominates the kinetic energy ( $\lambda > 0$ ) and the quasi-equilibrium state ceases to exist for a system in contact with a thermal bath (paper II). Notice that even in this case, the stable quasi-equilibrium state still exists for a system surrounded by an adiabatic wall. On the other hand, for more larger radius, while the curves with index  $n < 5$  abruptly terminate, the trajectories with  $n > 5$  next reach at another critical point  $d\eta/d\lambda = 0$ , i.e.,  $C_V = 0$ . Further, they progressively change their direction and finally spiral around a fixed point. The appearance of the critical point  $C_V = 0$  is explained as follows. While the inner part of the system keeps the specific heat negative, the outer part seems to have positive one. Thus, the heat current from inner to outer part causes the raise of the temperature at both parts. For a system with the radius  $r_e$  smaller than certain critical value, the amount of the heat capacity at outer part is small so that the outer part easily catches up with the increase of the inner-part temperature. As increasing  $r_e$ , the fraction of the outer normal part grows up and it eventually balances with the inner gravothermal part. Thus, beyond the point characterized by the condition  $C_V = 0$ , no thermal balance is attainable and the system becomes gravothermally unstable. This is even true in the system surrounded by an adiabatic wall.

From these discussions, one can immediately verify that the condition (48) represents the marginal stability for a system confined in an adiabatic wall,

which exactly coincides with the previous result using standard linear means. Hence, one concludes that the quasi-equilibrium structure obtained from the new formalism does not alter the thermal properties in micro-canonical ensemble case. On the other hand, the condition (48) indicates the onset of thermodynamic instability for a system in contact with a thermal bath (i.e., canonical ensemble case), which significantly differs from the results in old Tsallis formalism. Of course, this might be a natural consequence of the different choice of statistical average, leading to the different definition  $T_{\text{phys}}$ , however, the appearance of instability in present case might seem somewhat curious. While the previous results indicate the unstable state at the indices  $n > 3$ , consistent with the suggestion by Chavanis [28], Fig. 1 implies that thermodynamic instability appears for stellar polytrope with any value of the index  $n$ . One might worry about whether the present results rigorously match the stability analysis from the variational problem. In next section, to check the consistency of new Tsallis formalism, we develop the stability analysis based on the second variation of entropy and free-energy.

Finally, in Fig. 2, varying the radius  $r_e$ , the specific heat  $C_v$  is plotted as a function of density contrast,  $D \equiv \rho_c/\rho_e$  for typical polytrope indices with  $n \geq 3/2$ . Clearly, the critical point  $|C_v| \rightarrow \infty$  marked by crosses exists in each case, while the marginal stability  $C_v = 0$  only appears when  $n > 5$  at a certain high density contrast (c.f. Fig. 3 in paper II). The numerical values indicated by arrows represent the critical values  $D_{\text{crit}}$  evaluated at the point  $C_v = 0$ , which are the same results as in Table 1 of paper I (see also Fig. 5). These behaviors can also be deduced from the energy-radius-mass relation and the radius-mass-temperature relation. In Figs. 3 and 4, using the expressions (42) and (37), the dimensionless values  $\lambda$  and  $\eta$  for various polytrope indices are evaluated and plotted as a function of density contrast, respectively.<sup>3</sup> The Emden trajectories in  $(\eta, \lambda)$  plane (see Fig.1) state that the marginal stability for the system in contact with a thermal bath,  $|C_v| \rightarrow \infty$  implies the first turning point  $d\eta/d\xi_e = 0$ , or equivalently  $d\eta/dD = 0$ . Similarly, the marginal stability  $C_v = 0$  represents the condition  $d\lambda/dD = 0$ . From Figs. 3 and 4, we readily estimate the critical density contrast at the first turning points in each case, which exactly coincide with the points marked by the arrows and the crosses, respectively. Consistently, the turning point disappears when  $n < 5$  in Fig. 3, while it does always exist independently of the polytrope index in Fig. 4.

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<sup>3</sup> Fig. 3 is essentially the same result as in Fig. 2 of paper I.



#### 4 Stability/instability criteria from the variational problems

Previous section reveals the existence of two types of thermodynamic instability. Then, the marginal stability conditions (48) and (49) are obtained for a system confined in an adiabatic wall and for a system in contact with a thermal bath, respectively. In this section, in order to check the consistency of these results, we reconsider the marginal stability criteria based on the variational problem.

According to the maximum entropy principle, the stable quasi-equilibrium state for a system confined in an adiabatic wall is only possible when the second variation of entropy around the extremum state is negative, i.e.,  $\delta^2 S_q < 0$ . Similarly, the stable quasi-equilibrium distribution surrounded by a thermal bath should have minimum free-energy, indicating the positive value of the second variation of free-energy, i.e.,  $\delta^2 F_q > 0$ . Thus, the condition  $\delta^2 S_q = 0$  or  $\delta^2 F_q = 0$  readily implies the marginal stability in each case.

To begin with, let us write down the entropy of quasi-equilibrium state. From (24) and (28), we have

$$S_q = \left(n - \frac{1}{2}\right) \left\{ c_n \left( T_{\text{phys}}^{-(3/2)/(n-3/2)} W \right)^{-(n-3/2)/(n-1/2)} - 1 \right\}. \quad (50)$$

Here, just for convenience, we introduced the quantity  $W$ :

$$W = \int d^3x \rho^{1+1/n}. \quad (51)$$

Note that in deriving the equation (50), we used the relation  $T_{\text{phys}} = K_n W$  from (23).

Using the above expression (50), we compute the variation of entropy up to the second order terms. To be specific, we consider the density perturbation  $\delta\rho(r)$  around the quasi-equilibrium configuration, under keeping the total mass  $M$  and the radius  $r_e$  constant. Then the variation of entropy becomes

$$\begin{aligned} \delta S_q &= \left(n - \frac{1}{2}\right) \delta N_q \\ &= N_q \left[ \frac{3}{2} \frac{\delta T_{\text{phys}}}{T_{\text{phys}}} - n \frac{\delta W}{W} - \frac{3}{4} \frac{n-2}{n-1/2} \left( \frac{\delta T_{\text{phys}}}{T_{\text{phys}}} \right)^2 \right. \\ &\quad \left. + \frac{n(n-1/4)}{n-1/2} \left( \frac{\delta W}{W} \right)^2 - \frac{3}{2} \frac{n}{n-1/2} \frac{\delta T_{\text{phys}}}{T_{\text{phys}}} \frac{\delta W}{W} \right] \end{aligned} \quad (52)$$

As for the variation  $\delta W$ , we have

$$\delta W = \delta \left( \int d^3x \rho^{1+1/n} \right) = \frac{n+1}{n} \int d^3x \left\{ \delta \rho + \frac{1}{2n} \frac{(\delta \rho)^2}{\rho} \right\} \rho^{1/n}. \quad (53)$$

Also, we write down the variation of energy:

$$\begin{aligned} \delta E &= \delta \left( \frac{3}{2} \int d^3x P + \frac{1}{2} \int d^3x \rho \Phi \right) \\ &= \frac{3}{2} \delta T_{\text{phys}} + \frac{1}{2} \int d^3x (2 \Phi \delta \rho + \delta \rho \delta \Phi), \end{aligned} \quad (54)$$

where we used the fact that  $\int d^3x \rho \delta \Phi = \int d^3x \Phi \delta \rho$ .

Below, we separately analyze the variational problem in each case.

#### 4.1 *Stability/instability criterion from the second variation of entropy*

Let us first consider the stability/instability condition for a system confined in an adiabatic wall. In this case, the conservation of total energy  $E$  is always guaranteed and we further put another constraint  $\delta E = 0$ , which yields

$$\delta T_{\text{phys}} = -\frac{1}{3} \int d^3x (2 \Phi \delta \rho + \delta \rho \delta \Phi)$$

from (54). Substituting the above equation into (52), a straightforward calculation leads to the variation of entropy up to the second order, summarized as follows:

$$\delta S_q = \delta^{(1)} S_q + \delta^{(2)} S_q ;$$

$$\delta^{(1)} S_q = -\frac{N_q}{T_{\text{phys}}} \int d^3x \left\{ \Phi + (n+1) \frac{T_{\text{phys}}}{W} \rho^{1/n}(x) \right\} \delta \rho, \quad (55)$$

$$\begin{aligned} \delta^{(2)} S_q &= -N_q \left[ \int d^3x \left\{ \frac{1}{2 T_{\text{phys}}} \delta \rho \delta \Phi + \frac{1}{2W} \frac{n+1}{n} \rho^{1/n-1} (\delta \rho)^2 \right\} \right. \\ &\quad \left. + \frac{1}{3 T_{\text{phys}}^2} \frac{n-2}{n-1/2} \left( \int d^3x \Phi \delta \rho \right)^2 \right] \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{W^2} \frac{(n+1)^2(n-1/4)}{n(n-1/2)} \left( \int d^3x \delta\rho^{1/n} \delta\rho \right)^2 \\
& - \frac{1}{W T_{\text{phys}}} \frac{n+1}{n-1/2} \left( \int d^3x \rho^{1/n} \delta\rho \right) \left( \int d^3x \Phi \delta\rho \right) \Big]. \quad (56)
\end{aligned}$$

The resultant form of the second variation of entropy  $\delta^{(2)}S_q$  seems rather complicated, however, recalling the fact that the background solution of stellar polytropic distribution always satisfies the condition  $\delta^{(1)}S_q = 0$ , the last three terms in right-hand-side of equation (56) can be rewritten in more compact form. After some algebra, we obtain

$$\begin{aligned}
\delta^{(2)}S_q = & -N_q \left[ \int d^3x \left\{ \frac{1}{2T_{\text{phys}}} \delta\rho \delta\Phi + \frac{1}{2W} \frac{n+1}{n} \rho^{1/n-1} (\delta\rho)^2 \right\} \right. \\
& \left. + \frac{W}{3T_{\text{phys}}^2} \frac{n}{n-3/2} \left( \int d^3x \Phi + \frac{3}{2} \frac{n+1}{n} \frac{T_{\text{phys}}}{W} \delta\rho \right)^2 \right]. \quad (57)
\end{aligned}$$

Apart from the over-all positive constant, the above expression is indeed the same equation as previously obtained in paper I (see Eq.(36) in paper I, with the identification  $T_{\text{phys}}/W = \{(n-3/2)/(n+1)\}T$ ). Thus, just following the same calculation as in paper I, the stability/instability criterion from  $\delta^{(2)}S_q = 0$  can be obtained, from which one can rigorously prove that the marginal stability condition for a system confined within an adiabatic wall exactly reproduces the equation (48). As a result, for certain critical values of dimensionless energy and density,  $\lambda_{\text{crit}}$  and  $D_{\text{crit}}$ , the onset of the gravothermal instability appears when the stellar polytrope with index  $n > 5$  (see Figs.3 and 5).

#### 4.2 Stability/instability criterion from the second variation of free-energy

Now, turn to focus on the second variation of free-energy. In paper II, usual definition of free-energy

$$F'_q = E - \beta^{-1}S_q \quad (58)$$

has been used to investigate the marginal stability for a system surrounded by a thermal bath. As several authors recently pointed out, the usual definition of free-energy becomes inadequate when the physical temperature cannot be identified with the inverse of Lagrange multiplier  $\beta$ . According to the modification of Clausius relation (27), the free-energy must be generalized as follows

[24,25]:

$$F_q = E - T_{\text{phys}} \frac{1}{1-q} \ln \{1 + (1-q)S_q\}. \quad (59)$$

Using this *generalized* free-energy, we consider the variation up to the second order, keeping the thermodynamic temperature  $T_{\text{phys}}$  constant, not the total energy  $E$ :

$$\delta F_q = \delta E - T_{\text{phys}} \frac{1}{1-q} \left\{ \frac{\delta N_q}{N_q} - \frac{1}{2} \left( \frac{\delta N_q}{N_q} \right)^2 \right\}.$$

Substituting the equations (52), (53) and (54) into the above equation and keeping the terms up to the second order, after some manipulation, one obtains

$$\delta F_q = \delta^{(1)} F_q + \delta^{(2)} F_q ;$$

$$\delta^{(1)} F_q = \int d^3 x \left\{ \Phi(x) + (n+1) \frac{T_{\text{phys}}}{W} \rho^{1/n}(x) \right\} \delta \rho, \quad (60)$$

$$\begin{aligned} \delta^{(2)} F_q = \frac{1}{2} \int d^3 x \left\{ \delta \rho \delta \Phi + \frac{n+1}{n} \frac{T_{\text{phys}}}{W} \rho^{1/n-1} (\delta \rho)^2 \right\} \\ - \frac{1}{2} \frac{T_{\text{phys}}}{n} \left( \frac{n+1}{W} \right)^2 \left( \int d^3 x \rho^{1/n} \delta \rho \right)^2. \end{aligned} \quad (61)$$

Note that apart from the over-all factor, the first variation  $\delta^{(1)} F_q$  exactly coincides with the first variation of entropy,  $\delta^{(1)} S_q$ , which ensures the fact that extremum state of generalized free-energy is the same quasi-equilibrium distribution as obtained from the maximum entropy principle. Using the fact  $\delta^{(1)} F_q = 0$ , equation (60) is now rewritten with

$$\begin{aligned} \delta^{(2)} F_q = \frac{1}{2} \int d^3 x \left\{ \delta \rho \delta \Phi + \frac{n+1}{n} \frac{T_{\text{phys}}}{W} \rho^{1/n-1} (\delta \rho)^2 \right\} \\ - \frac{1}{n T_{\text{phys}}} \left( \int d^3 x \Phi \delta \rho \right)^2. \end{aligned} \quad (62)$$

Obviously, the above equation differs from the old result based on the usual definition of free-energy (58) (c.f. Eq.(51) in paper II). This fact readily implies that the thermodynamic stability has nothing to do with the dynamical stability, which apparently contradicts with the recent claim by Chavanis [27,28].

Thus, in present case using normalized  $q$ -values, it seems non-trivial whether the above equation indeed leads to the marginal stability condition (49).

In what follows, restricting our attention to the radial mode of density and potential perturbations, we discuss the existence or the absence of perturbation mode satisfying the equation  $\delta^{(2)}F_q = 0$ . Following the papers I and II, we introduce the new perturbed quantity:

$$\delta\rho(r) \equiv \frac{1}{4\pi r^2} \frac{dQ(r)}{dr}. \quad (63)$$

Then the mass conservation  $\delta M = 0$  implies the boundary condition  $Q(0) = Q(r_e) = 0$ . Substituting (63) into (62) and repeating the integration by parts, the second variation of generalized free-energy is reduced to the following quadratic form:

$$\delta^{(2)}F_q = -\frac{1}{2} \int_0^{r_e} dr_1 \int_0^{r_e} dr_2 Q(r_1) \hat{K}(r_1, r_2) Q(r_2), \quad (64)$$

where  $\hat{K}(r_1, r_2)$  stands for the operator given by

$$\begin{aligned} \hat{K}(r_1, r_2) \equiv & \left\{ \frac{n+1}{n} \frac{d}{dr_1} \left( \frac{1}{4\pi r_1^2 \rho(r_1)} \frac{P(r_1)}{\rho(r_1)} \frac{d}{dr_1} \right) + \frac{G}{r_1^2} \right\} \delta_D(r_1 - r_2) \\ & + \frac{1}{nT_{\text{phys}}} \frac{d\Phi(r_1)}{dr_1} \frac{d\Phi(r_2)}{dr_2}. \end{aligned} \quad (65)$$

Thus, the problem reduces to the eigenvalue problem and the marginal stability condition just corresponds to the zero-eigenvalue problem:

$$\int_0^{r_e} dr' \hat{K}(r, r') Q(r') = 0, \quad (66)$$

which yields

$$\begin{aligned} \hat{L}[Q] \equiv & \left[ \frac{n+1}{n} \frac{d}{dr} \left( \frac{1}{4\pi r^2 \rho} \frac{P}{\rho} \frac{d}{dr} \right) + \frac{G}{r^2} \right] Q \\ & = -\frac{1}{nT_{\text{phys}}} \frac{G m(r)}{r^2} \int_0^{r_e} dr' \frac{G m(r')}{r'^2} Q(r'). \end{aligned} \quad (67)$$

To obtain the solution of the above zero-eigenvalue problem, we follow the similar procedure in paper I. First recall the fact that the following equations are satisfied:

$$\hat{L}[4\pi r^3 \rho(r)] = \frac{n-3}{n} \frac{Gm(r)}{r^2}, \quad \hat{L}[m(r)] = \frac{n-1}{n} \frac{Gm(r)}{r^2}. \quad (68)$$

Then, these relations allow us to put the ansatz of the solution,

$$Q(r) = c_1 4\pi r^3 \rho(r) + c_2 m(r), \quad (69)$$

and to determine the coefficients  $c_1$  and  $c_2$  by substituting the ansatz into (67):

$$\frac{n-3}{n} c_1 + \frac{n-1}{n} c_2 = \Lambda_n, \quad (70)$$

where we define

$$\Lambda_n \equiv -\frac{1}{nT_{\text{phys}}} \int_0^{r_e} dr \frac{Gm(r)}{r^2} Q(r). \quad (71)$$

In addition to the above condition, we must further consider the boundary conditions  $Q(0) = Q(r_e) = 0$ . While the condition  $Q(0) = 0$  is automatically fulfilled, the remaining condition  $Q(r_e) = 0$  requires

$$c_1 4\pi r_e^3 \rho_e + c_2 M = 0. \quad (72)$$

Hence, from (70) and (72), the coefficients  $c_1$  and  $c_2$  are determined and are expressed in terms of homology invariants:

$$c_1 = \frac{n \Lambda_n}{n-3-(n-1)u_e}, \quad c_2 = -\frac{n u_e \Lambda_n}{n-3-(n-1)u_e}. \quad (73)$$

As noted in paper I, the above coefficients are not specified completely because of the quantity  $\Lambda_n$  in the coefficients, which depends on the solution (69) itself. To eliminate this self-referential structure, we directly evaluate the non-local term  $\Lambda_n$ , leading to the consistency condition for the existence of solution (69). Substitution of the solution (69) with the coefficients (73) into the expression (71) yields

$$1 = -\frac{T_{\text{phys}}^{-1}}{n-3-(n-1)u_e} \int_0^{r_e} dr \frac{Gm(r)}{r^2} \{4\pi r^3 \rho(r) - u_e m(r)\}.$$

In the above equation, the integrals in the right-hand-side can be evaluated using the formulae (89) and (90) listed in Appendix B. Further, the radius-mass-temperature relation (42) eliminates the dependence of  $T_{\text{phys}}$ , leading to

$$1 = \frac{v_e}{2u_e + v_e - (n+1)} \frac{1}{n-3 - (n-1)u_e} \times \left[ (n-5)u_e + (2u_e - 1) \left\{ 3 + (n+1) \left( \frac{u_e}{v_e} - \frac{3}{v_e} \right) \right\} \right].$$

After some algebra, the above equation is rewritten with the following quadratic form:

$$\begin{aligned} 0 &= 4u_e^2 + 2u_e v_e - (n+9)u_e - v_e + n + 1 \\ &= k(u_e, v_e), \end{aligned} \tag{74}$$

which coincides with the marginally stability criterion (49) derived from the condition for specific heat,  $C_v \rightarrow \infty$ .

Therefore, unlike the first impression, we reach at the fully satisfactory conclusion that the thermodynamic instability inferred from the specific heat is consistently explained from the variational problems. In other words, the application of new Tsallis formalism to the stability of stellar quasi-equilibrium system reveals a consistent thermodynamic structure of self-gravitating system, as well as the existence of thermodynamic instability.

## 5 Discussion & conclusion

In this paper, we revisited the issues on the thermodynamic properties of stellar self-gravitating system arising from the Tsallis entropy, with a particular emphasis to the standard framework using the normalized  $q$ -values. It then turns out that the new extremum-entropy state essentially remains unchanged from the previous study and is characterized by the stellar polytrope, although the distribution function shows several distinct properties. Taking these facts carefully, the thermodynamic temperature of the extremum state was identified through the modified Clausius relation and the specific heat was evaluated explicitly. The detailed discussion on the behavior of specific heat finally leads to the conclusion that the stability of the system surrounded by a thermal wall(canonical case) is drastically changed from previous result, while the onset of gravothermal instability remains unchanged for a system confined in an adiabatic wall(micro-canonical case). The existence of these thermodynamic instabilities can also be deduced from the variation of entropy and free-energy

rigorously. As a result, above the certain critical values of  $\lambda$  or  $D$ , the thermodynamic instability appears at  $n > 5$  for a system confined in an adiabatic wall. As for the system in contact with a thermal bath, the onset of thermodynamic instability appears at  $(\eta_{\text{crit}}, D_{\text{crit}})$  with any value of  $n$ . In Figs. 5 and 6, for the sake of the completeness, the critical values evaluated from the conditions (48) and (49) are summarized, respectively. Note also the fact that the critical values in Fig. 5 exactly coincide with the previous results (c.f. Table 1 in paper I).

The one noticeable point in the present result is that the macroscopic relation such as the radius-mass-temperature relation or the specific heat as a function of density contrast can be solely determined from the Emden solutions without any uncertainty. Indeed, previous study using the standard linear means seriously suffer from the residual dimensional parameter  $h = (l_0 v_0)^3$ , which must be practically disappeared from the macroscopic description. In paper II, the origin of this residual dependence has been addressed in connection with the non-extensivity of the entropy. Although the scaling relation appeared in the old results is simply deduced from the asymptotic behavior of the Emden solutions, the explicit  $h$ -dependence itself has originated from the radius-mass-temperature relation. By contrast, in present case, the radius-mass-temperature relation was derived from the non-trivial relation (23), in which no such  $h$ -dependence appears. The resultant specific heat is thus obtained free from the residual dependence, which is a natural outcome of the new framework using the normalized  $q$ -values. Therefore, it seems likely that the new formalism provides a better characterization for non-extensive meta-equilibrium state.

In fact, for several application of the new formalism, the equation of state for non-extensive system has been found to be similar to the result from the ordinary extensive thermodynamics. For instance, the equation of state for the classical gas model without interaction reduces to the ideal gas state even in the power-law nature of velocity distribution [24]. Indeed, this fact can be clearly seen in our case neglecting the gravity. Recalling that both the density and the pressure become *homogeneous* in the limit  $G \rightarrow 0$ , the equation (41) immediately reduces to the relation  $T_{\text{phys}} = PV$ , i.e.,  $P \propto \rho T_{\text{phys}}$ , where  $V$  denotes the volume of the system. This result apparently seems contradiction with the polytropic relation (19), however, it turns out that the dimensional constant  $K_n$  immediately yields the relation  $K_n = P/\rho^{1+1/n}$  from the definition (21) and thereby the polytropic relation makes no sense in the limit  $G \rightarrow 0$ . Thus, the stellar polytropic system using normalized  $q$ -values successfully recovers the ideal gas limit, in contrast to the old results (see Appendix B in paper II). In other words, if the interaction is turned on, the equation of state for the ideal gas is no longer valid and instead the polytropic relation holds. In Sec.2, the polytropic relation was derived without assuming any specific choice of the gravitational potential. This in turn suggests that at



least in the mean-field treatment, the polytropic relation is a common feature in presence of long-range interaction and it generally holds for non-extensive quasi-equilibrium system.

Finally, we have firmly confirmed that the stellar polytropic system can be consistently characterized as a plausible meta-equilibrium state in the new framework of the non-extensive thermostatics, as well as in the old formalism. At present, however, one cannot rigorously discriminate the correct and the applicable formalism among them (there might be another possibility that both the formalism become indeed correct depending on the situations). While the new framework by means of the normalized  $q$ -values has theoretically desirable properties as mentioned above, the results in old formalism might provide an interesting connection with dynamical instability of gaseous system, as has been suggested by Chavanis [27,28]. In paper II, the thermodynamic instability of stellar polytropic system in the canonical ensemble case is shown to exist at the polytrope indices  $n > 3$ , which is exactly the same condition as derived from the gaseous system. On the other hand, in the new formalism, there is no such correspondence, since the instability is completely different from that of the gaseous system (see Sec.4.2). This means that the choice of the statistical average is crucial for the thermodynamic properties and the physical reason for this discrepancy should be further clarified. In any case, in order to pursue the physical reality of the non-extensive thermostatics, the limitation of thermodynamical approach is now apparent and the detailed kinematical study based on the Boltzmann or the Fokker-Planck equation provides an deep insight into the thermal transport property. In the light of this, the long-term stellar dynamical evolution by  $N$ -body simulation also becomes useful and the analysis is now in progress. The results will be presented elsewhere.

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## Appendix A: Extremum-state of Tsallis entropy from the OLM method

Here, owing to the maximum Tsallis entropy principle, we derive an extremum state of the stellar quasi-equilibrium distribution by means of the OLM method. According to Ref.[17], with a help of the relation (8), the variational problem that extremizes the entropy under the energy constraint becomes

$$\delta \left[ S_q - \alpha \left\{ \int d^6 \boldsymbol{\tau} p - 1 \right\} - \beta \int d^6 \boldsymbol{\tau} \left\{ M \left( \frac{1}{2} v^2 + \frac{1}{2} \Phi \right) - E \right\} p^q \right] = 0. \quad (75)$$

Then the variation with respect to probability function  $p(\mathbf{x}, \mathbf{v})$  leads to

$$\begin{aligned} \int d^6 \boldsymbol{\tau} \left[ -\frac{1}{q-1} (q p^{q-1} - 1) - \alpha - \beta M \left( \frac{1}{2} v^2 - \frac{E}{M} \right) q p^{q-1} \right] \delta p \\ - \frac{1}{2} \beta \delta \left( N_q \int d^6 \boldsymbol{\tau} \Phi f \right) = 0. \end{aligned} \quad (76)$$

In the above equation, the last term in the left-hand-side is rewritten with

$$\begin{aligned} \delta \left( N_q \int d^6 \boldsymbol{\tau} \Phi f \right) &= \delta N_q \int d^6 \boldsymbol{\tau} \Phi f + 2 N_q \int d^6 \boldsymbol{\tau} \Phi \delta f \\ &= q \int d^6 \boldsymbol{\tau} \left\{ 2M \Phi - \int d^6 \boldsymbol{\tau}' \Phi(\mathbf{x}') f(\mathbf{x}', \mathbf{v}') \right\} p^{q-1} \delta p, \end{aligned} \quad (77)$$

with a help of the equation (11). In the last line, we have exchanged the role of the variables  $(\mathbf{x}, \mathbf{v}) \leftrightarrow (\mathbf{x}', \mathbf{v}')$ . Substituting (77) into (76), we arrive at

$$\int d^6 \boldsymbol{\tau} \left[ -\frac{1}{q-1} (q p^{q-1} - 1) - \alpha - \beta M q p^{q-1} \left( \frac{1}{2} v^2 + \Phi - \varepsilon \right) \right] \delta p = 0,$$

where the quantity  $\varepsilon$  is given by (13). Recalling the fact that the above equation holds independently of the choice of the variation  $\delta p$ , we obtain

$$-\frac{1}{q-1} (q p^{q-1} - 1) - \alpha - \beta M q p^{q-1} \left( \frac{1}{2} v^2 + \Phi - \varepsilon \right) = 0,$$

which leads to the power-law distribution. The resultant expressions for the one-particle distribution function (escort distribution) is summarized in equations (15) and (16), with the identification  $N'_q = 1$ .

## Appendix B: Stellar polytropic system characterized by Emden solutions

In this appendix, we briefly describe the equilibrium configuration of stellar polytrope in the spherically symmetric case, together with some useful formulae which has been used in the main analysis of section 3 and 4.

First notice that the one-particle distribution function (15) does not yet completely specify the equilibrium configuration, due to the presence of gravitational potential which implicitly depends on the distribution function itself. Hence, we need to specify the gravitational potential or density profile. From the gravitational potential (3), it reads

$$\frac{1}{r^2} \frac{d}{dr} \left\{ r^2 \frac{d\Phi(r)}{dr} \right\} = 4\pi G \rho(r). \quad (78)$$

Combining the above equation with (17), we obtain the ordinary differential equation for  $\Phi$ . Alternatively, a set of equations which represent the hydrostatic equilibrium are derived using (78), (17) and (18):

$$\frac{dP(r)}{dr} = -\frac{Gm(r)}{r^2} \rho(r), \quad (79)$$

$$\frac{dm(r)}{dr} = 4\pi \rho(r) r^2. \quad (80)$$

The quantity  $m(r)$  denotes the mass evaluated at the radius  $r$  inside the wall. Denoting the central density and pressure by  $\rho_c$  and  $P_c$ , we then introduce the dimensionless quantities:

$$\rho = \rho_c [\theta(\xi)]^n, \quad r = \left\{ \frac{(n+1)P_c}{4\pi G \rho_c^2} \right\}^{1/2} \xi, \quad (81)$$

which yields the following ordinary differential equation:

$$\theta'' + \frac{2}{\xi} \theta' + \theta^n = 0, \quad (82)$$

where prime denotes the derivative with respect to  $\xi$ . The quantities  $\rho_c$  and  $P_c$  in (81) are the density and the pressure at  $r = 0$ , respectively. To obtain the physically relevant solution of (82), we put the following boundary condition:

$$\theta(0) = 1, \quad \theta'(0) = 0. \quad (83)$$

A family of solutions satisfying (83) is referred to as the *Emden solution*, which is well-known in the subject of stellar structure (e.g., see Chap.IV of Ref.[21]).

To characterize the equilibrium properties of Emden solutions, it is convenient to introduce the following set of variables, referred to as homology invariants [21,22]:

$$u \equiv \frac{d \ln m(r)}{d \ln r} = \frac{4\pi r^3 \rho(r)}{m(r)} = -\frac{\xi \theta^n}{\theta'}, \quad (84)$$

$$v \equiv -\frac{d \ln P(r)}{d \ln r} = \frac{\rho(r)}{P(r)} \frac{Gm(r)}{r} = -(n+1) \frac{\xi \theta'}{\theta}, \quad (85)$$

which reduce the degree of equation (82) from two to one. The derivative of these variables with respect to  $\xi$  becomes

$$\frac{du}{d\xi} = \left(3 - u - \frac{n}{n+1} v\right) \frac{u}{\xi}, \quad \frac{dv}{d\xi} = \left(-1 + u + \frac{1}{n+1} v\right) \frac{v}{\xi}. \quad (86)$$

Equations (82) can thus be re-written with

$$\frac{u}{v} \frac{dv}{du} = \frac{(n+1)(u-1) + v}{(n+1)(3-u) - nv}. \quad (87)$$

The corresponding boundary condition to (83) becomes  $(u, v) = (3, 0)$ .

Now, utilizing the above homology invariants, we list some useful formulae which can be derived from the hydrostatic equations (79) and (80) (see also Appendix A in paper I):

$$\begin{aligned} \int_0^{r_e} dr 4\pi r^2 P(r) &= -\frac{1}{n-5} \left\{ 8\pi r_e^3 P_e - (n+1) \frac{M P_e}{\rho_e} + \frac{G M^2}{r_e} \right\} \\ &= -\frac{1}{n-5} \frac{G M^2}{r_e} \left( 2 \frac{u_e}{v_e} - \frac{n+1}{v_e} + 1 \right), \end{aligned} \quad (88)$$

$$\begin{aligned} \int_0^{r_e} dr \frac{Gm(r)}{r^2} 4\pi r^3 \rho(r) &= -4\pi r_e^3 P_e + 3 \int_0^{r_e} dr 4\pi r^2 P(r), \\ &= -\frac{n+1}{n-5} \frac{G M^2}{r_e} \left\{ \frac{u_e}{v_e} - \frac{3}{v_e} + \frac{3}{n+1} \right\}, \end{aligned} \quad (89)$$

$$\begin{aligned}
\int_0^{r_e} dr \frac{G m^2(r)}{r^2} &= -\frac{GM^2}{r_e} - 8\pi r_e^3 P_e + 6 \int_0^{r_e} dr 4\pi r^2 P(r), \\
&= -\frac{n+1}{n-5} \frac{GM^2}{r_e} \left\{ 2 \left( \frac{u_e}{v_e} - \frac{3}{v_e} \right) + 1 \right\}.
\end{aligned} \tag{90}$$

Finally, using these formulae, we evaluate the total energy of the stellar system in terms of the variables at the edge  $r_e$ :

$$\begin{aligned}
E = K + U &= \frac{3}{2} \int_0^{r_e} dr 4\pi r^2 P(r) - \int_0^{r_e} dt \frac{Gm(r)}{r} \frac{dm}{dr} \\
&= -\frac{1}{n-5} \left[ \frac{3}{2} \left\{ \frac{GM^2}{r_e} - (n+1) \frac{MP_e}{\rho_e} \right\} + (n-2) 4\pi r_e^3 P_e \right],
\end{aligned}$$

which can be re-expressed in terms of the homology invariants:

$$E = \frac{1}{n-5} \frac{GM^2}{r_e} \left[ \frac{3}{2} \left\{ 1 - \frac{n+1}{v_e} \right\} + (n-2) \frac{u_e}{v_e} \right]. \tag{91}$$

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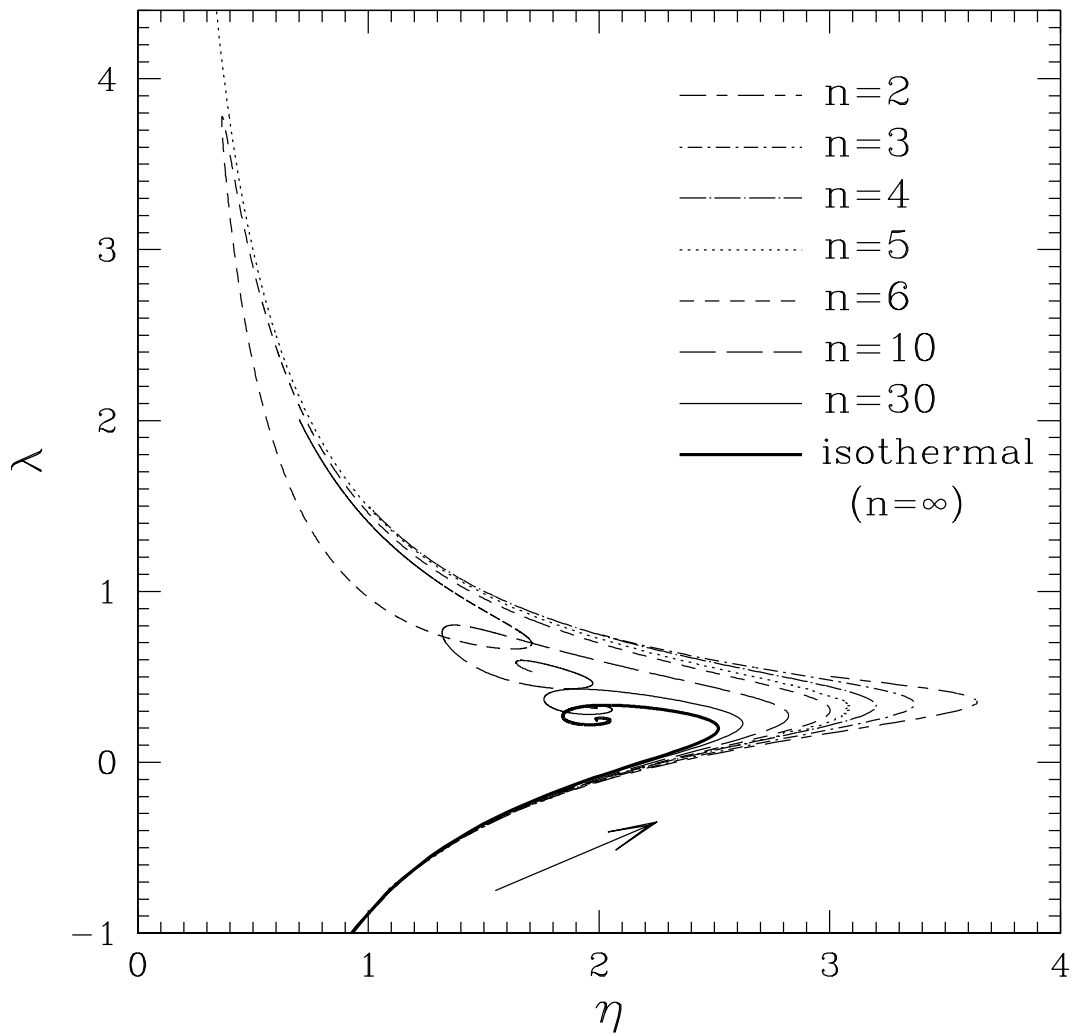


Fig. 1. Trajectory of Emden solutions in  $(\eta, \lambda)$ -plane.



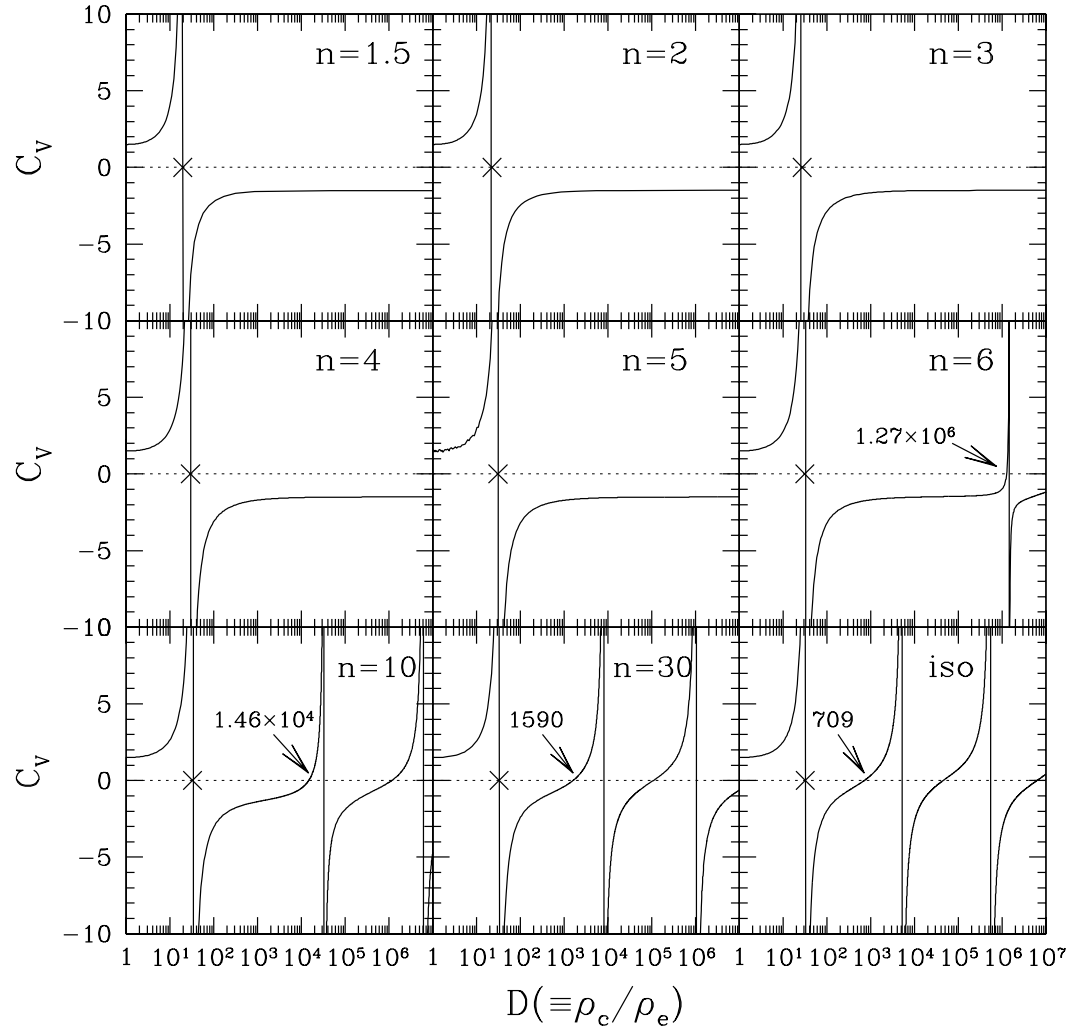


Fig. 2. Specific heat as a function of density contrast  $D(= \rho_c/\rho_e)$  for various polytropic indices.

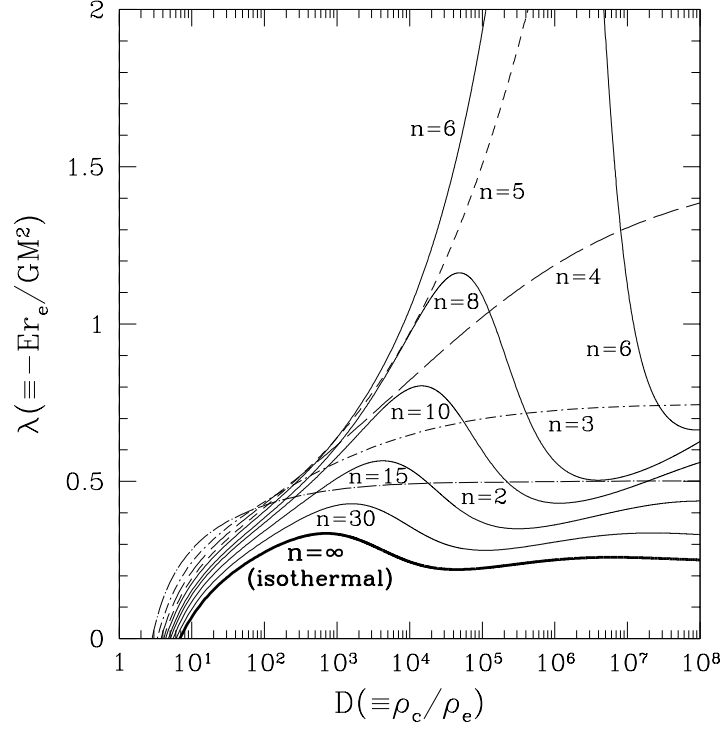


Fig. 3. Energy-radius-mass relation as a function of density contrast.

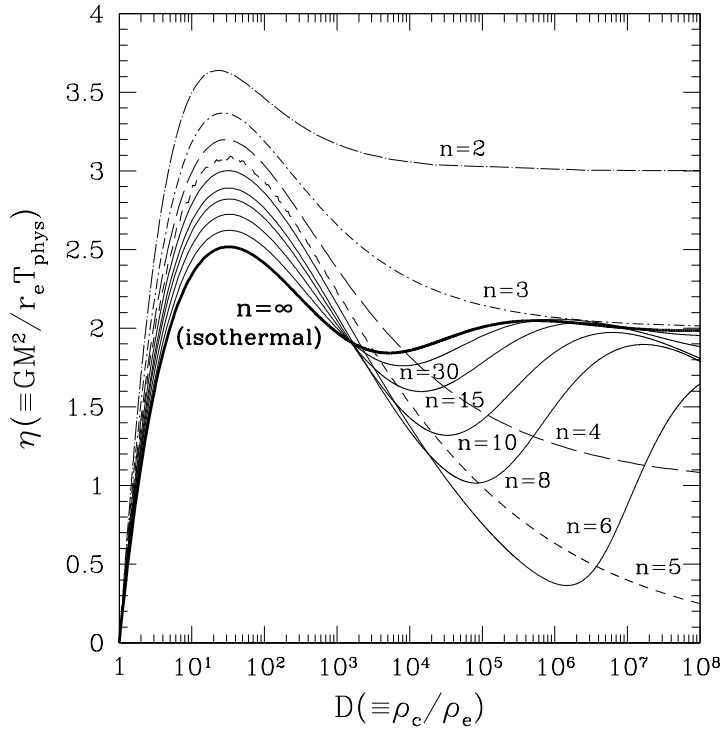


Fig. 4. Radius-mass-temperature relation as a function of density contrast.

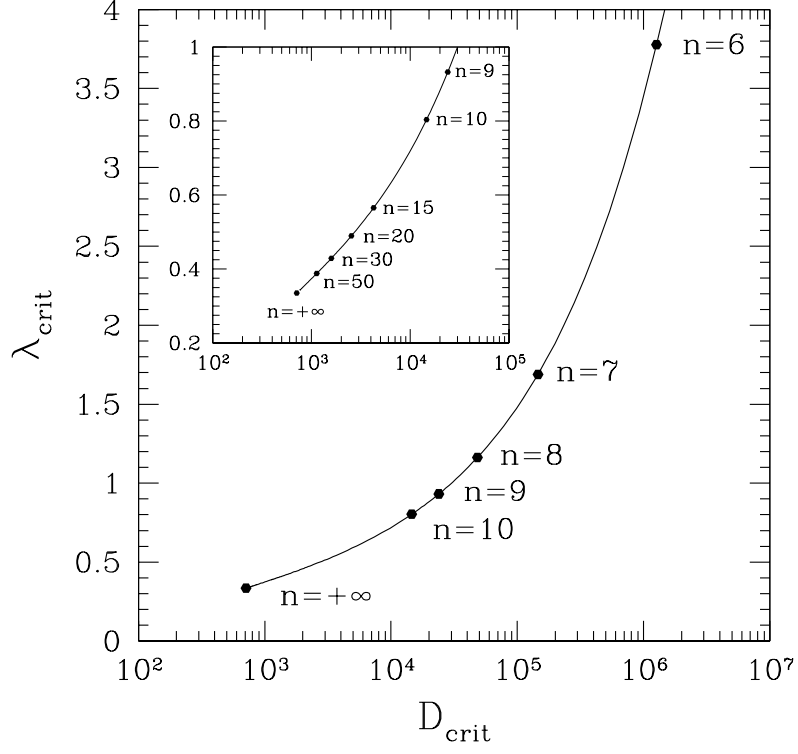


Fig. 5. Critical values  $(D_{\text{crit}}, \lambda_{\text{crit}})$  for a system confined in an adiabatic wall.

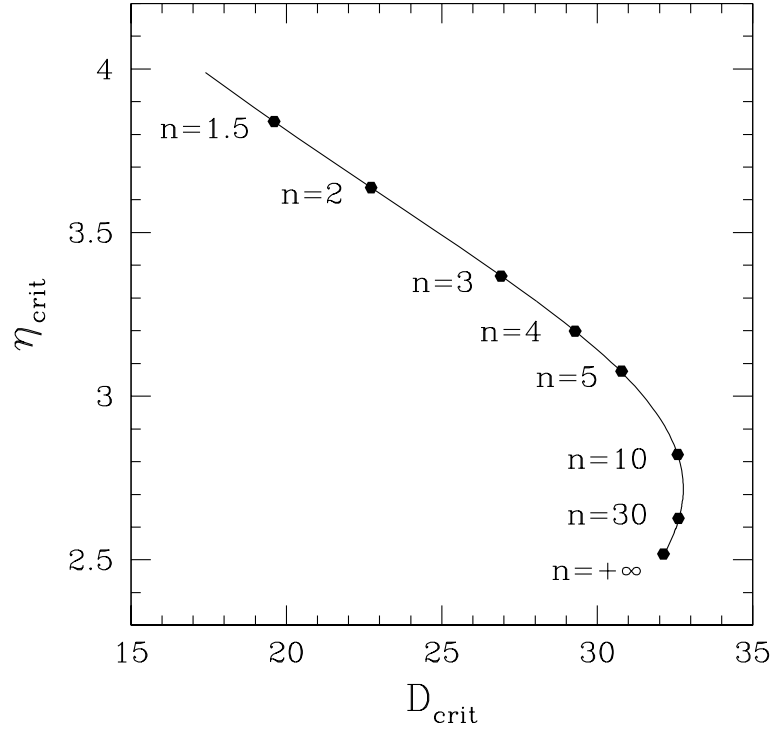


Fig. 6. Critical values  $(D_{\text{crit}}, \eta_{\text{crit}})$  for a system in contact with a thermal bath.